

LS-152
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Analysis of a Third-Order Sum Resonance

It is worth considering an experiment on a sum resonance. I will give an analytic treatment of a third-order sum resonance. The treatment parallels that in LS-132 for the Walkinshaw difference resonance. Although the algebra is essentially the same as for the difference resonance, the sum resonance appears to have a richer structure.

1. Analysis of the Resonance.

The Hamiltonian in the neighborhood of the sum resonance

$$\nu_x + 2\nu_y = m \quad (1.1)$$

can be written in terms of angle-action variables in the form

$$h = \nu_x J_x + \nu_y J_y + S(2J_x)^{1/2} (2J_y) \sin(\delta_x + 2\delta_y - m\theta + \xi) \\ + a J_x^2 + 2b J_x J_y + c J_y^2. \quad (1.2)$$

We first transform to resonant coordinates via the generating function

$$F(J_1, J_2, \delta_x, \delta_y, \theta) = J_1(\delta_x + 2\delta_y - m\theta + \xi) - J_2 \delta_y, \quad (1.3)$$

which gives

$$\gamma_1 = \delta_x + 2\delta_y - m\theta + \xi, \quad \gamma_2 = \delta_y, \quad (1.4)$$

$$J_x = J_1, \quad J_y = 2J_1 - J_2, \quad (1.5)$$

$$J_1 = J_x, \quad J_2 = 2J_x - J_y. \quad (1.6)$$

The resonant hamiltonian is

$$h_s = \epsilon J_1 - \nu_y J_2 + 2S(2J_1)^{1/2}(2J_1 - J_2) \sin \gamma_1 \\ + a J_1^2 + 2b J_1(2J_1 - J_2) + c (2J_1 - J_2)^2, \quad (1.7)$$

where

$$\epsilon = \nu_x + 2\nu_y - m. \quad (1.8)$$

We see that J_2 is a constant of the motion. If $J_2 > 0$, i.e. if $J_y < 2J_x$, then the motion is required by Eq.(1.4) to lie outside the circle $2J_1 = J_2$ in the J_1, γ_1 -phase plane. Recall that for the difference resonance, the motion was required to lie inside this circle. In contrast to the difference resonance, for large initial y-amplitudes near the sum resonance, J_2 can be negative, and the circle $2J_1 = J_2$ does not exist. Moreover, as we see from Eq.(1.6), the amplitudes J_x, J_y must increase and decrease together!

In rectangular coordinates

$$Q = (2J_1)^{1/2} \sin \gamma_1, \quad P = (2J_1)^{1/2} \cos \gamma_1, \quad (1.9)$$

the hamiltonian is

$$h_s = \frac{1}{2} \epsilon (P^2 + Q^2) + 2SQ(P^2 + Q^2 - J_2) \\ + \frac{1}{4} a (P^2 + Q^2)^2 + b (P^2 + Q^2)(P^2 + Q^2 - J_2) + c (P^2 + Q^2 - J_2)^2, \quad (1.10)$$

where the irrelevant constant - $v_y J_2$ has been omitted.

For the value $h_s = (1/2)\epsilon J_2 + (1/4)aJ_2^2$ corresponding to the limiting circle $J_2 - 2J_1 = 0$, Eq.(1.10) factors into the product of two circles:

$$(P^2 + Q^2 - J_2) \left[\frac{1}{2}\epsilon + 2SQ + \frac{1}{4}a(P^2 + Q^2 + J_2) + b(P^2 + Q^2) + c(P^2 + Q^2 - J_2) \right] = 0 \quad (1.11)$$

The first factor is the limiting circle, and the second is the circle

$$P^2 + \left(Q + \frac{S}{A}\right)^2 = \frac{S^2}{A^2} - \frac{B}{A} \quad , \quad (1.12)$$

where

$$A = \frac{1}{4}a + b + c \quad , \quad B = \frac{1}{2}\epsilon + \left(\frac{1}{4}a - c\right)J_2 \quad . \quad (1.13)$$

Note that except for a few sign differences, formulas (1.10), (1.11) and (1.12) are the same as the corresponding formulas in LS-132. We will call the limiting circle $2J_1 = J_2$ the "small" circle, and the circle (1.12) the "large" circle. The names are descriptive if the nonlinearities are very small. However, if they are large, then the large circle will disappear if $B/A > S^2/A^2$. The circles intersect (if at all) in the points

$$Q_0 = -\frac{B + AJ_2}{2S} = -\frac{\epsilon + (a + 2b)J_2}{4S} \quad , \quad P_0 = \pm (J_2 - Q_0^2)^{1/2} \quad . \quad (1.14)$$

There are now a number of cases, depending on whether either or both circles exist, whether they intersect, and if not, whether they lie outside each other or whether one lies inside the other, as well as on the number and position of the various fixed points of the motion. A number of typical cases are sketched in Fig. 1. The small circle is shaded, since its interior is a non-physical region. The large circle is drawn with a dashed line. Since both the interior and the exterior of the large circle are accessible, the existence or non-existence of the large circle has no effect on the topology of the constant h_s contours, unless the large circle intersects the small circle. In all cases, the motion is stable at large amplitudes, because when $P^2 + Q^2$ is very large, Eq.(1.10) becomes the equation of a circle.

There are two reasons for the richer variety for the sum resonance. First, the motion lies outside the small circle, and the contour plot of h_s may have more structure there.

Second, for the difference resonance, we assumed small nonlinearities, which rules out many of the cases in Fig. 1. For the sum resonance, both x and y motions go to very large amplitudes, if we start above the coupling threshold and the nonlinearities are small. Therefore, if we are to do the experiment without losing beam every time we exceed the threshold, we must depend on the nonlinearities to confine the motion. We must therefore consider finite nonlinearities in analyzing the sum resonance, as we did for the third-integral resonance $3\nu_x = m$. In case the initial y-amplitude is very small, J_2 is always positive, and the number of cases we must consider is smaller.

It is convenient both experimentally and theoretically to consider motions in which the y-amplitude is initially very small and the x-amplitude is finite. In that case, $J_2 = x_0^2$, where x_0 is the initial x-amplitude. The only cases that can occur are those labelled a,...,e in Fig.1. The motion lies on a contour which is close to the small circle. Case a is the unstable case, and occurs for $x_0^2 > Q_0^2$, i.e., for

$$x_1 < x_0 < x_2, \quad (1.15)$$

where

$$x_1^2, x_2^2 = \frac{8S^2}{(a+2b)^2} \left\{ 1 \pm \left[1 - \frac{\epsilon(a+2b)}{4S^2} \right]^{\frac{1}{2}} \right\} - \frac{\epsilon}{a+2b} \quad (1.16)$$

For small nonlinearities, the threshold and upper limit are given by

$$x_1 \doteq \left| \frac{\epsilon}{4S} \right|, \quad x_2 \doteq \left| \frac{4S}{a+2b} \right|, \quad \text{if } \frac{\epsilon(a+2b)}{4S^2} \ll 1. \quad (1.17)$$

The maximum x and y amplitudes are

$$x_m = Q_m, \quad y_m = \left[2(Q_m^2 - x_0^2) \right]^{\frac{1}{2}}, \quad (1.18)$$

where

$$Q_m = \left| \frac{S}{A} \right| \left\{ 1 + \left[1 - \frac{AB}{S^2} \right]^{\frac{1}{2}} \right\} \xrightarrow[\frac{AB}{S^2} \rightarrow 0]{} 2 \left| \frac{S}{A} \right|. \quad (1.19)$$

The last expression is an approximation for small nonlinearities. It will be important to check whether parameters can be found such that the maximum y-amplitude is within the vacuum chamber, in order to make an experiment on the sum resonance

practical. For small nonlinearities, and x_0 at the threshold value, the maximum y - amplitude is

$$y_m \doteq 2\sqrt{2} \left| \frac{S}{A} \right| - \frac{\epsilon^2 |A|}{128\sqrt{2} S^2} \quad (1.20)$$

If S/A can be made small enough, the experiment is practical.

In cases b and c in Fig.1, the y motion is stable at small amplitudes, (contours near the small circle), but there is a threshold for the y-amplitude above which the motion is unstable.

Just above the threshold for the difference resonance, the growth in y-amplitude is very small. In contrast, as we see from Fig.1a, just above the threshold for the sum resonance, (when the circles overlap only slightly), the y-amplitude will grow to a very large value, along with the x-amplitude. The threshold for the sum resonance is abrupt and dramatic. It is possible that even if the maximum y-amplitude is outside the vacuum chamber, we might be able to do the experiment if we are careful to exceed the threshold only slightly. We would then have only a fraction of the beam above threshold, and would lose only a fraction of it in each experiment.

If nonlinearities are significant at the difference resonance, then we can also have both a threshold and an upper limit for the resonant coupling, as in Eq.(1.16) above.

2. Connection with the Real Ring.

The calculation in Section 2 of LS-132 gives all terms up to third order, including the term which drives the sum resonance. All formulas in LS-132 apply also here. The sextupole coefficient in the resonant term in Eq.(1.2) is given by Eq.(2.4) of LS-132:

$$S_s = \left(\frac{\beta_x^{1/2} \beta_y B'' l}{16 \pi B \rho} \right)_{s=s_j} \quad (2.1)$$

The amplitude S_s of the sextupole term that drives the sum resonance is exactly the same as the amplitude S_w that drives the difference resonance. The phase is, from Eq.(2.5), LS-132:

$$\zeta = m s_j / R - \psi_{xj} - 2 \psi_{yj} \quad (2.2)$$

These formulas give the amplitude and phase of the driving term for a single sextupole. If there are more than one sextupole, each contributes a resonance term in Eq.(1.2), and the terms must be added.

3. Transforming the Non-Resonant Terms.

The non-resonant terms in the complete hamiltonian can be transformed away by the method used in LS-132, Section 3. The calculation is precisely the same, except that the term which is dropped corresponds to the sum resonance (1.1). The resulting formulas for the corrections to the coefficients in Eq.(1.2) are, corresponding to Eqs.(3.5)-(3.7) in LS-132:

$$a_s = 6 S_3^2 \sum_{m'} \left[\frac{3}{m' - 3\nu_x} + \frac{1}{m' - \nu_x} \right] = 6 S_3^2 \left[-\frac{1}{\delta\nu_x} - \frac{3}{\delta(3\nu_x)} + 2\delta\nu_x + 6\delta(3\nu_x) \right], \quad (3.1)$$

$$b_s = \sum_{m'} \left\{ \frac{24 S_3 S_5 - 8 S_5^2}{\nu_x - m'} - \frac{2 S_5^2}{\nu_x + 2\nu_y - m'} - \frac{2 S_5^2}{\nu_x - 2\nu_y - m'} \right\} \\ = (24 S_3 S_5 - 8 S_5^2) \left[\frac{1}{\delta\nu_x} - 2\delta\nu_x \right] - 2 S_5^2 \left[\frac{1}{\delta(\nu_x - 2\nu_y)} - 2\delta(\nu_x - 2\nu_y) - 2\delta(\nu_x + 2\nu_y) \right], \quad (3.2)$$

$$c_s = -12 S_5^2 \sum_{m'} \left[\frac{4}{\nu_x - m'} + \frac{1}{\nu_x + 2\nu_y - m'} + \frac{1}{\nu_x - 2\nu_y - m'} \right] \\ = -12 S_5^2 \left[\frac{4}{\delta\nu_x} + \frac{1}{\delta(\nu_x - 2\nu_y)} - 8\delta\nu_x - 2\delta(\nu_x + 2\nu_y) - 2\delta(\nu_x - 2\nu_y) \right], \quad (3.3)$$

where primes on the summations mean that the resonant term is to be omitted, and where S_3 is given by Eq.(2.12) of LS-131:

$$S_3 = \left(\frac{\beta_x^{3/2} B'' l}{48 \pi B \rho} \right)_{s=s_j} . \quad (3.4)$$

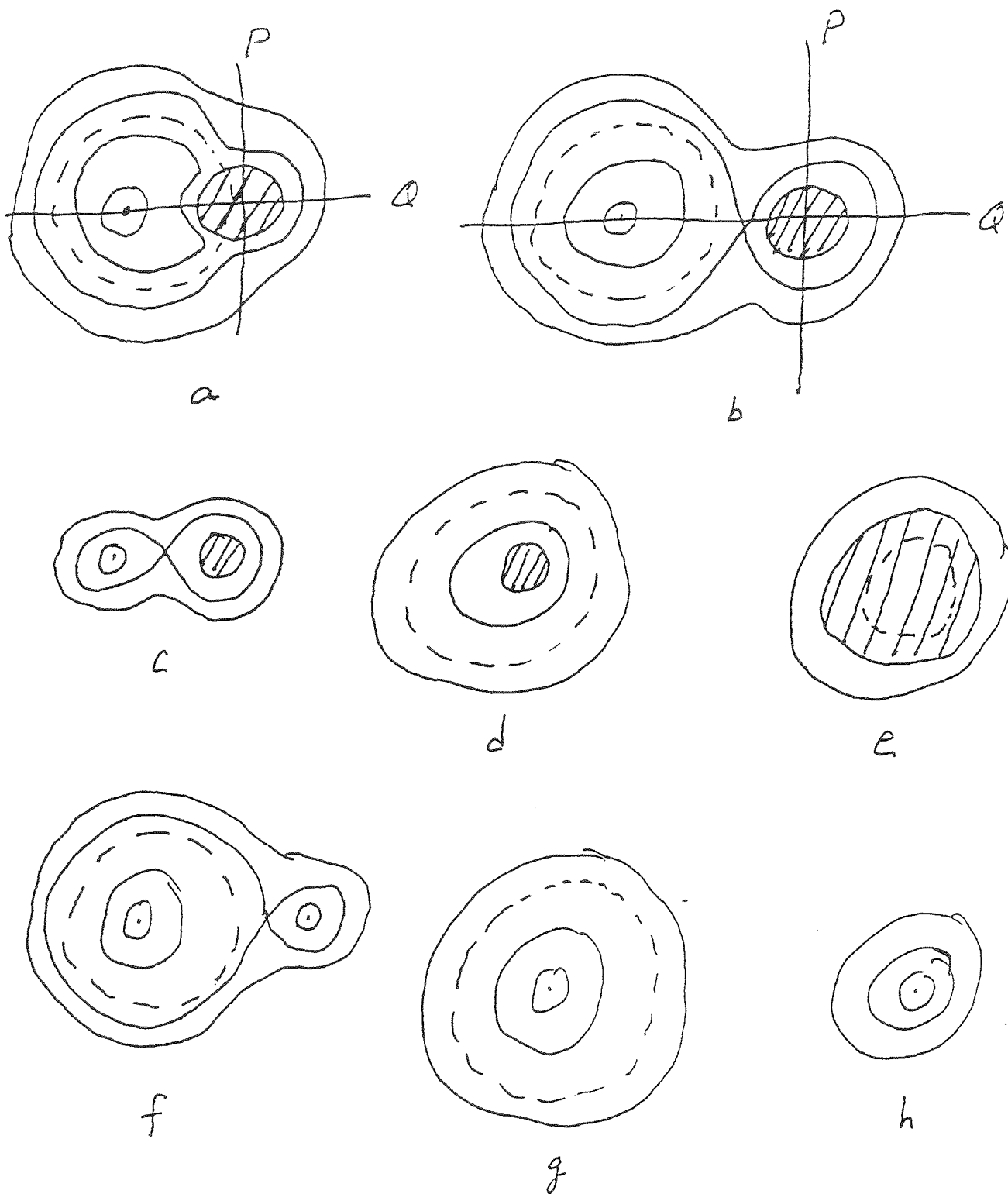


Figure 1. Contour sketches of h_s for various parameter ranges.